Density fluctuations and phase transition in the Nagel-Schreckenberg traffic flow model

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We consider the transition of the Nagel-Schreckenberg traffic flow model from the free flow regime to the jammed regime. We examine the inhomogeneous character of the system by introducing a new method of analysis which is based on the local density distribution. We investigated the characteristic fluctuations in the steady state and present the phase diagram of the system.
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I. INTRODUCTION

Over the past few years much attention has been devoted to the study of traffic flow. Since the seminal work of Lighthill and Whitham in the middle of the 50's [1] many attempts have been made to construct more and more sophisticated models which incorporate various phenomena occurring in real traffic (for an overview see [2]). Recently, a new class of models, based on the idea of cellular automata, has been proven to describe traffic dynamics in a very efficient way [3]. Especially the transition from free flow to jammed traffic with increasing car density could be investigated very accurately. Nevertheless, besides various indications [4], no unique description for a dynamical transition could be found. Furthermore, no satisfying order parameter could be defined so far. In this article we introduce a new method of analysis which allows us to identify the different phases of the system and to describe the phase transition in detail, i.e., defining an order parameter, considering the fluctuations which drive the transition, and determining the phase diagram.

We consider a one-dimensional cellular automaton of linear size L and N particles. Each particle is associated the integer values $v_i \in \{0,1,2,...,v_{max}\}$ and $d_i \in \{0,1,2,3,...\}$, representing the velocity and the distance to the next forward particle [3]. For each particle, the following four update steps representing the acceleration, the slowing down, the noise, and the motion of the particles are done in parallel: (1) if $v_i < d_i$ then $v_i \rightarrow \mathrm{Min}\{v_i+1, v_{max}\}$, (2) if $v_i > d_i$ then $v_i \rightarrow d_i$, (3) with probability P $v_i \rightarrow \mathrm{Max}\{v_i - 1, 0\}$, and (4) $r_i \rightarrow r_i + v_i$, where r_i denotes the position of the i-th particle.

II. SIMULATIONS AND RESULTS

Figure 1 shows a space-time plot of the system. Each dot corresponds to a particle at a given time step. The global density $\rho_g = N/L$ exceeds the critical density and jams occur. Traffic jams are characterized by a high local density of the particles and by a backward movement of shock waves [1]. One can see from Fig.1 that in the jammed regime the system is inhomogeneous, i.e., traffic jams with a high local density and free flow regions with a low local density coexist. In order to investigate this transition one has to take this inhomogeneity into account.

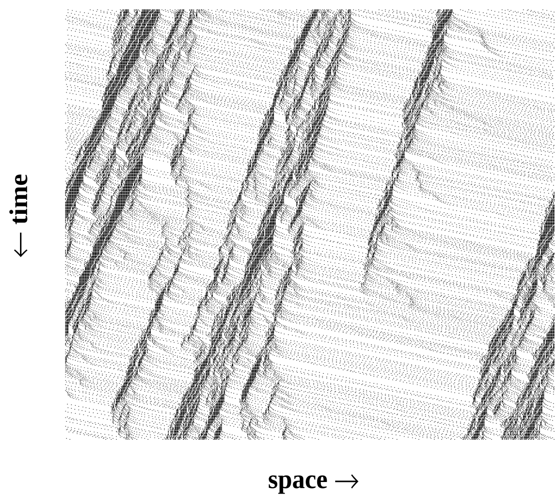

FIG. 1. Space-time plot for $v_{max} = 5$, $P = \frac{1}{2}$, and $\rho_g > \rho_c$. Note the separation of the system in high and low density regions.

Traditionally one determines the so-called fundamental diagram, i.e., the diagram of the flow vs the density. The global flow is given by, $\Phi = \rho_g \langle v \rangle$, where $\langle v \rangle$ denotes the averaged velocity of the particles. Due to the stochastic behavior of the dynamics for $0 < P < 1$ the system behaves independently of the initial conditions after a certain transient regime. In this limit one can interpret $\langle ... \rangle$ as a time or ensemble average. For $P = 0$ and $P = 1$ the dynamics is deterministic and the behavior depends strongly on the initial conditions. In any case, this non-local measurements are not sensitive to the inhomogeneous character of the system, i.e., the information about the two different coexisting phases is lost. In the following we apply a new method of analysis which is based on the measurement of the local density distribution $p(\rho)$. The local density ρ is measured on a section of the system of size δ according to

$$\rho \;=\; \frac{1}{\rho_g \delta} \sum_{i=1}^{N} \theta(\delta - r_i). \tag{1}$$

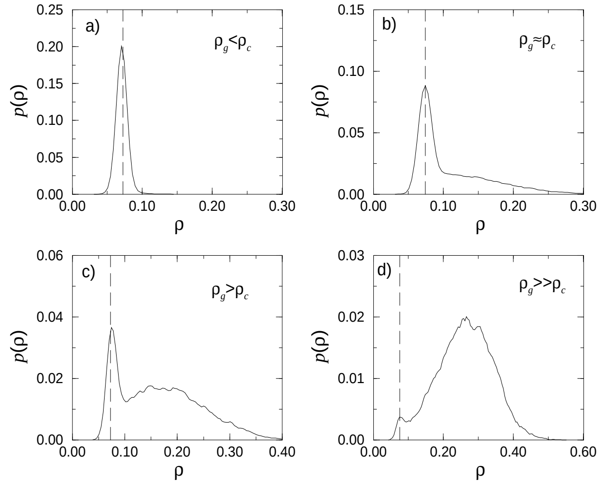

FIG. 2. The local density distribution $p(\rho)$ for various values of the global density, $v_{max} = 5$, $P = \frac{1}{2}$ and $\delta = 256$. The dashed line corresponds to the characteristic density of the free flow phase.

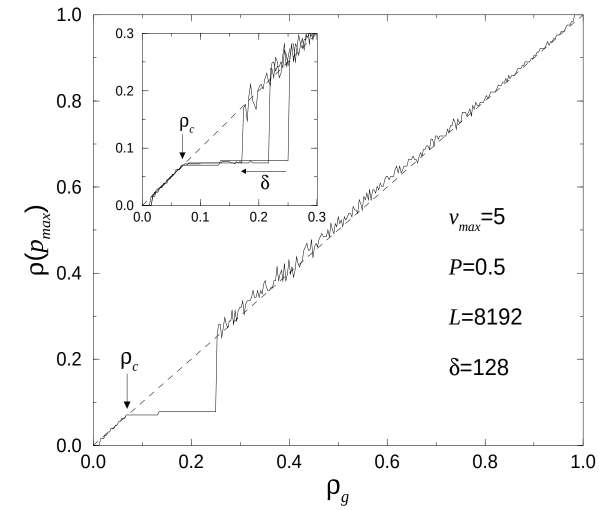

FIG. 3. The maximum of the local density distribution as a function of the global density ρ_g. The inset shows that the transition point ρ_c does not depend on the value of the parameter δ.

Of course we have checked that the main results are not affected by the value of δ, provided that delta is significantly smaller than the system size L in order to measure the local properties. In order to reflect the behavior of the low density regime δ should be significantly larger than a certain length scale λ_0 which corresponds to the characteristic length scale of the density fluctuations in the free flow phase (see below). For any parameter set $\{v_{max}, P\}$ the local density ρ fluctuates around the value of the global density ρ_g and the probability distribution of the local density $p(\rho)$ contains all informations needed to describe the transition.

The local density distribution $p(\rho)$ is plotted for various values of the global density ρ_g in Fig. 2. In the case of small values of ρ_g, see Fig. 2a, the particles can be considered as independent (see below) and the local density distribution is simply Gaussian with the mean values ρ_g and a width which scales with $\sqrt{\delta}$. Increasing the global density, jams occur and the distribution displays two different peaks (Fig. 2c). The first peak corresponds to the density of free particles and in the phase coexistence regime the position of this peak does not depend on the global density (see the dashed lines in Fig. 2). The second peak is located at larger densities and characterizes the jammed phase. With increasing density the second peak occurs in the vicinity of the critical density ρ_c (Fig. 2b) and grows further (Fig. 2c) until it dominates the distribution in the sense that the first peak disappears (Fig. 2d). The two peak structure of the local density distribution clearly reflects the coexistence of the free flow and jammed phase above the critical value ρ_c. In the following we show that the behavior of the peaks leads to a determination of ρ_c.

One expects that in a homogeneous system the local density distribution displays one peak and is symmetric around the global density, i.e., $\rho(p_{max}) = \rho_g$. This cannot be the case in an inhomogeneous system where the local density distribution displays two peaks corresponding to two different coexisting phases. In Fig. 3 we plot the position of the maximum of the local density distribution $\rho(p_{max})$ as a function of the global density ρ_g. One clearly sees the transition point ρ_c where the position of the maximum becomes independent of the global density. The inset of Fig. 3 shows that the determination of the transition point does not depend on the special value of δ. Only the point where the second peak exceeds the first peak depends on the measurement parameter δ. With increasing δ this point tends to smaller values of ρ_g because with increasing δ the measurement starts to average over the two different phases. From these measurements we conclude that the phase transition of the Nagel-Schreckenberg model is a transition from a homogeneous regime (free flow phase) to an inhomogeneous regime which is characterized by a coexistence of two phases (free flow traffic and jammed traffic).

In order to describe the spatial decomposition of the coexisting phases we measured the steady state structure factor [5]

$$S(k) \; = \; \frac{1}{L} \left\langle \left| \sum_{r=1}^{L} \eta(r)\, e^{ikr} \right|^2 \right\rangle, \qquad (2)$$

where $\eta(r) = 1$ if the lattice site r is occupied and $\eta(r) = 0$ otherwise. In Fig. 4 we plot the structure factor $S(k)$ for the same values of the global density as in Fig. 2, i.e., below, in the vicinity, above and far away of the transition point. It is remarkable that $S(k)$ exhibits a maximum for all considered values of the global density at $k_0 \approx 0.72$ (dashed lines in Fig. 4). This value correspondence to the characteristic wave length $\lambda_0 = \frac{2\pi}{k_0}$ of the density fluctuations in the free flow phase. The steady state structure factor is related to the Fourier

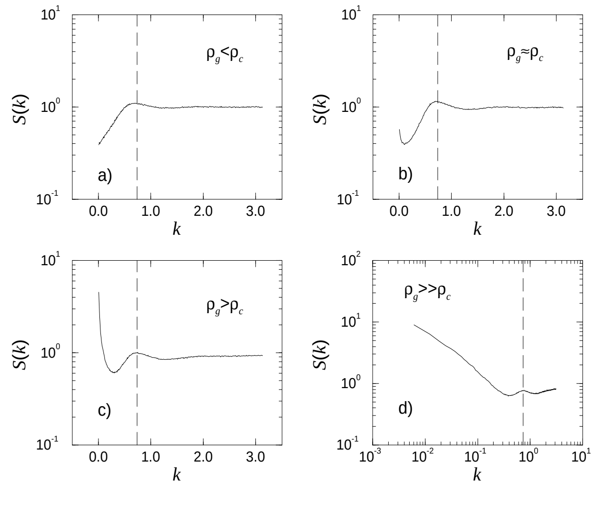

FIG. 4. The structure factor $S(k)$ for $P = \frac{1}{2}$, $v_{max} = 5$ and for various values of the density ρ. The dashed line marks the characteristic wavelength λ_0 of the free flow phase.

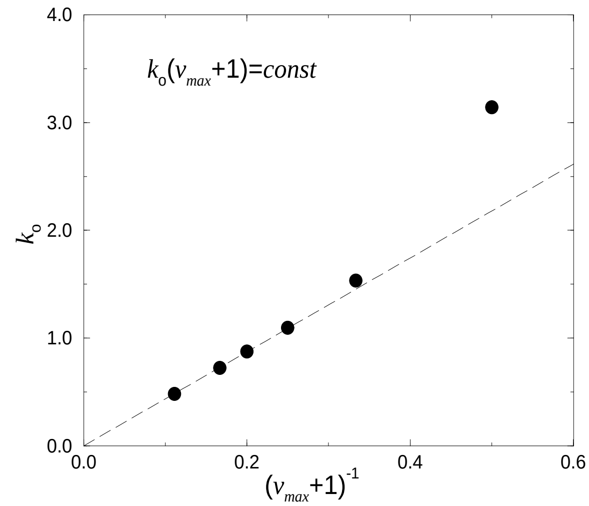

FIG. 5. The characteristic inverse wavelength k_0 for the maximum velocities $v_{max} \in \{1, 2, 3, 4, 5, 8\}$ for $P = \frac{1}{2}$.

transform of the real space density-density correlation function. The wave length λ_0 corresponds to a maximum of the correlation function, i.e., λ_0 describes the most likely distance of two particles in the free flow phase. For low densities the structure factor is almost independent of the density and displays a minimum for small k values indicating the lack of long-range correlations. Crossing the transition point the smallest mode $S(k = \frac{2\pi}{L})$ increases quickly. This suggests that the jammed phase is characterized by long-range correlations which decay in the limit $\rho_g \gg \rho_c$ algebraically as one can see from the log-log plot in Fig. 4d.

We already mentioned that k_0 characterizes the density fluctuations in the free flow phase. In Fig. 5 we plot k_0 as a function of v_{max}. Except for the case $v_{max} = 1$ the maximum velocity and k_0 obey the relation $k_0\,(v_{max} + 1) = const$. The fact that $v_{max} = 1$ violates this equation does not surprise. It is already known that the physics for $v_{max} \geq 2$ is distinctly different from the case $v_{max} = 1$ [6]. For instance, for $v_{max} = 1$ the fundamental diagram is symmetric around its maximum at $\rho_g = 0.5$ independent of the noise parameter P. Whereas, the position of the maximum depends on P for $v_{max} \geq 2$ and no symmetry occurs. Another example is that for $v_{max} \geq 2$ jams are allowed to branch (see Fig. 1), unlike jams for $v_{max} = 1$ [7]. The qualitative different behavior for $v_{max} = 1$ is caused by a particle-hole symmetry which is lost for larger values of the maximum velocity [6].

Changing from momentum space to real space the characteristic wavelength λ_0 of the density fluctuations in the free flow phase are given by

$$\lambda_0 \;=\; \frac{2\pi}{k_0} \;=\; 2\pi\,\frac{v_{max} + 1}{const}. \tag{3}$$

On the other hand the average distance $\bar{d}$ of the particles is given by the inverse density ρ_g^{-1}. In the free flow phase the average distance is larger than the wavelength λ_0, $\lambda_0 \ll \bar{d} = 1/\rho_g$, i.e., the cars can be considered as independent particles. With increasing density this behavior changes when the average distance $\bar{d}$ is comparable to the wavelength $\lambda_0 \approx \bar{d}$. The critical density is related to the characteristic wavelength

$$\rho_c \;=\; \frac{1}{\bar{d}} \;\approx\; \frac{1}{\lambda_0} \;\sim\; \frac{1}{v_{max} + 1}. \tag{4}$$

The fact that the critical density scales with $v_{max} + 1$ is already known for the deterministic case $P = 0$ [8].

Up to now we only considered the case $P = \frac{1}{2}$. The phase diagram in Fig. 6 shows the P dependence of the transition density ρ_c. f denotes the free flow phase and f+j corresponds to the coexistence region where the system separates in the free flow and jammed phase. The dashed line displays the P dependence of the maximum flow obtained from an analysis of the fundamental diagram [9]. The critical densities ρ_c, where the phase transition takes place, are lower than the density values of the maximum flow. Measurements of the relaxation time, which is expected to diverge at a transition point [4], confirm this result [10] (see Fig. 6). But one has to mention that the determination of the critical density via relaxation times leads in the coexistence regime f+j to unphysical results, in the sense that the relaxation time becomes negative [10].

Motivated by real traffic flow we fixed in our measurements the noise parameter P and varied the global density ρ_g, i.e., we crossed the critical line in the phase diagram parallel to the vertical axis (see Fig. 6). In theoretical investigations however, it is more convenient and revealing to consider the crossing of the critical line parallel to the horizontal axis, i.e., increasing the noise parameter P by a fixed density. With growing P density fluctuations with a characteristic wavelength λ_0 occur. This wavelength λ_0 grows with increasing P but no phase separation takes place until it exceeds at the transition line a critical wavelength λ_c. Then the system separates into

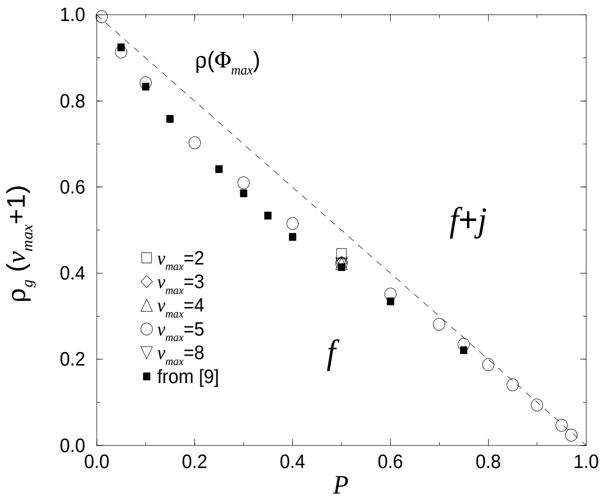

FIG. 6. The phase diagram of the Nagel-Schreckenberg model. Note that in the non-deterministic region $0 < P < 1$ the density of the maximum flow exceeds the density of the transition point.

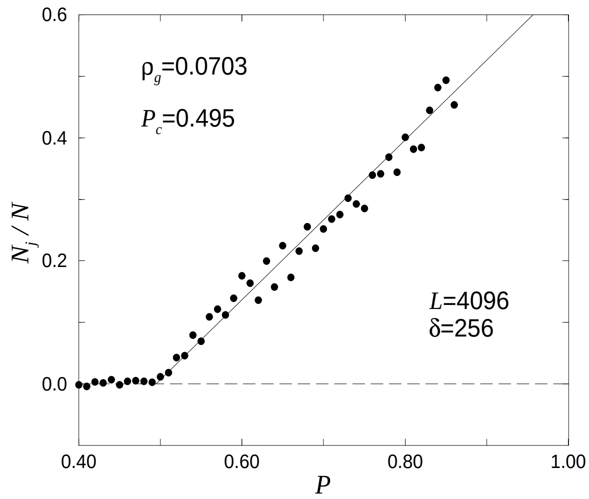

FIG. 7. The number of particles in the high density phase N_j normalized by the total number of particles N. N_j/N could serve as an order parameter and the critical exponent is $\beta = 1$ (see solid line).

the two coexisting phases and the amount of particles which belong to the jammed phase, N_j, could serve as an order parameter. As mentioned above the local density distribution displays two peaks in the heterogeneous phase and the area under each peak is proportional to N_f and N_j, respectively. Figure 7 displays N_j normalized by the total number of particles ($N = N_f + N_j$). Approaching the transition point p_c, N_j arises linear, i.e., it obeys the equation $N_j \sim (P - P_c)^\beta$, where the critical exponent is given by $\beta = 1$. The continuously behavior of N_j and the diverging relaxation time [10] suggest that the transition of the traffic model could describe as a phase transition of second order. A detailed analysis of the order parameter and the order parameter fluctuations, including a finite-size analysis, requests further investigations.

III. CONCLUSIONS

In conclusion we have studied numerically the Nagel-Schreckenberg traffic flow model using a local density analysis. Crossing the critical line of the system a phase transition takes place from a homogeneous regime (free flow phase) to an inhomogeneous regime which is characterized by a coexistence of two phases (free flow traffic and jammed traffic). The decomposition in the phase coexistence regime is driven by density fluctuations, provided they exceed a critical wavelength λ_c. The amount of particles in the jammed phase could serve as an order parameter which arises linear at the transition point, suggesting that the transition is of second order.

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